CHAPTER 5: UNCERTAINTY IN EXPERIMENT

Introduction

An experimental result has no physical meaning unless an uncertainty (or error) is assigned to it. Getting the wrong answer when multiplying or dividing numbers is not an "error", but a mistake and is something which you should recognize and be able to correct.

The error is not found by comparing your answer to some number in the textbook: uncertainties cannot be avoided in experimental physics. Multiple measurements of the same quantity using the same measuring instrument, may not give the same result each time due to random errors. If a systematic error is present, a set of very precise measurements may miss the true value completely. For example, a stopwatch is able to measure times to within 0.01 s, but there is often an associated error due to reaction time of about 0.3 s.

Why error needs to be evaluated?

The concept of error needs to be well understood. What is and what is not meant by "error"?

A measurement may be made of a quantity that has an accepted value, which can be looked up in a handbook (e.g.. the density of brass). The difference between the measurement and the accepted value is **not** what is meant by error. Such accepted values are not "right" answers. They are just measurements made by other people which have errors associated with them as well.

Nor does error mean "blunder." Reading a scale backwards, misunderstanding what you are doing or elbowing your lab partner's measuring apparatus are blunders which can be caught and should simply be disregarded.

Obviously, it cannot be determined exactly how far off a measurement is; if this could be done, it would be possible to just give a more accurate, corrected value.

Error, then, has to do with uncertainty in measurements that nothing can be done about. If a measurement is repeated, the values obtained will differ and none of the results can be preferred over the others. Although it is not possible to do anything about such error, it can be characterized. For instance, the repeated measurements may cluster tightly together or they may spread widely. This pattern can be analyzed systematically.

Source and classification of an error

A knowledge of errors is a first step in finding way to reduce them. Errors may arise from various sources. No measurement can be made with perfectness, but it is significant to find out how different errors have entered into the measurement so that they can be taken into consideration.

There are number of ways in which the measurement errors can be classified. However, we will discuss here only three important classification.

The first classification is on the basis of the manner in which the errors affect the results:

- independent error – error which add directly and in no circumstances compensate each other in calculation of the result e.g. creeping of errors in measurement of height and diameter in calculation or the volume of prismatic cylinder
- dependent error – error which could possibly compensate and reduce or even nullify the individual error e.g. using incorrect deflecting coil and incorrect permanent magnet in an ammeter
- correlated error – due to functional relation existing between the dependent and independent errors e.g. a poor temperature compensated strain gauge is used during elevated temperature strain testing wherein temperature and strain measurement are both done.

The second method classifies measurement errors into two broad types:

- controllable errors – errors the cause of which are definable as to kind and magnitude; such errors can be controlled that is they can be determined and use in computation
- incidental errors – not controllable because they cannot be determined and used in computation; since their causes cannot be stated in numerical terms, it appears as if these errors were not casually definable for which reason they are called incidental.

The third method classifies the measurement errors into three types:

- gross / illegimate errors
- systematic / fixed errors
- random / residual errors

Gross errors are mainly due to human mistakes in reading measurement, recording and calculating measurement results. It can be avoided by careful reading and recording the data. Two or more reading can be taken by different persons at different reading point to avoid re-reading with the same error.

Systematic errors are errors that tend to shift all measurements in a systematic way so their mean value is displaced. This may be due to such things as incorrect calibration of equipment, consistently improper use of equipment or failure to properly account for some effect. In a sense, a systematic error is rather like a blunder and large systematic errors can and must be eliminated in a good experiment. But small systematic errors will always be present. For instance, no instrument can ever be calibrated perfectly.

Other sources of systematic errors are external effects which can change the results of the experiment, but for which the corrections are not well known. In science, the reasons why several independent confirmations of experimental results are often required (especially using different techniques) is because different apparatus at different places may be affected by different systematic effects. Aside from making mistakes (such as thinking one is using the x10 scale, and actually using the x100 scale), the reason why experiments sometimes yield results, which may be far outside the quoted errors, is because of systematic effects that were not accounted for.

In summary, systematic error can be divided into three categories:

- instrument errors – these errors are due to the admitted tolerances of the various components of the measuring instruments or to faulty adjustment in assembling it; there are three types of instrument errors:
    - due to inherent shortcomings of instruments – inherent in instrument because of their mechanical structure
    - due to misuse of instruments – due to the fault of the experimenter than that of the instrument; may be failure to adjust the zero of instruments, poor initial adjustments, using leads of too high a resistance and so on.
    - due to loading effect – are committed by beginners by the improper use of an instruments for measurement work
- environmental errors – due to influence of environment and surroundings; some technique to reduce this errors:
    - conditions may be kept as constant as possible
    - using equipment that is immune to these effects
    - using technique that eliminate the effects of these disturbances
    - efforts should be made to avoid the use of application of computed corrections, but where these corrections are needed and are necessary, they are incorporated for the computational of the results.
- observational errors – caused by faulting observation of the measurement.

Random errors are errors that fluctuate from one measurement to the next. They yield results distributed about some mean value. They can occur for a variety of reasons.

- They may occur due to lack of sensitivity. For a sufficiently a small change an instrument may not be able to respond to it or to indicate it or the observer may not be able to discern it.
- They may occur due to noise. There may be extraneous disturbances which cannot be taken into account.
- They may be due to imprecise definition.
- They may also occur due to statistical processes such as the roll of dice.

Random errors displace measurements in an arbitrary direction whereas systematic errors displace measurements in a single direction. Some systematic error can be substantially eliminated (or properly taken into account). Random errors are unavoidable and must be lived with.

Many times you will find results quoted with two errors. The first error quoted is usually the random error, and the second is called the systematic error. If only one error is quoted, then the errors from all sources are added together. (In quadrature as described in the section on propagation of errors.)

A good example of "random error" is the statistical error associated with sampling or counting. For example, consider radioactive decay which occurs randomly at a some (average) rate. If a sample has, on average, 1000 radioactive decays per second then the expected number of decays in 5 seconds would be 5000. A particular measurement in a 5 second interval will, of course, vary from this average but it will generally yield a value **within** 5000 +/- . Behavior like this, where the error,

$$\Delta n = \sqrt{n_{expected}}\text{, (1)}$$

is called a Poisson statistical process. Typically if one does not know $n_{expected}$ it is assumed that,

$$n_{measured} = n_{expected},$$

in order to estimate this error.

Determining random errors.

How can we estimate the uncertainty of a measured quantity? Several approaches can be used, depending on the application.

(a) Instrument Limit of Error (ILE) and Least Count

The **<u>least count</u>** is the smallest division that is marked on the instrument. Thus a meter stick will have a least count of 1.0 mm, a digital stopwatch might have a least count of 0.01 sec.

The **<u>instrument limit of error, ILE</u>** for short, is the precision to which a measuring device can be read, and is always equal to or smaller than the least count. Very good measuring tools are calibrated against standards maintained by the National Institute of Standards and Technology.

The Instrument Limit of Error is generally taken to be the least count or some fraction (1/2, 1/5, 1/10) of the least count). You may wonder **which to choose**, the least count or half the least count, or something else. No hard and fast rules are possible; instead you must be guided by common sense. If the space between the scale divisions is large, you may be comfortable in estimating to 1/5 or 1/10 of the least count. If the scale divisions are closer together, you may only be able to estimate to the nearest 1/2 of the least count, and if the scale divisions are very close you may only be able to estimate to the least count.

For some devices the ILE is given as a tolerance or a percentage. Resistors may be specified as having a tolerance of 5%, meaning that the ILE is 5% of the resistor's value.

Problem: For each of the following scales (all in centimeters) determine the least count, the ILE, and read the length of the gray rod.

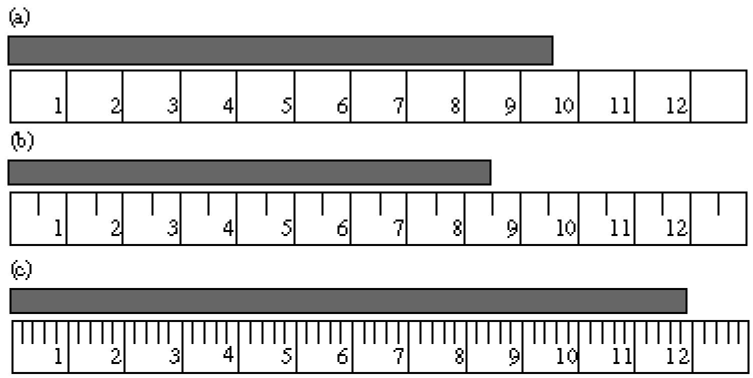


Figure 1 For each object and scale above, determine the least count of the scale, the ILE, and the length of the gray rod. The scales are all in centimeters.

Instrument Limit of Error: Answer

Problem: For each of the following scales determine the least count, and the ILE.

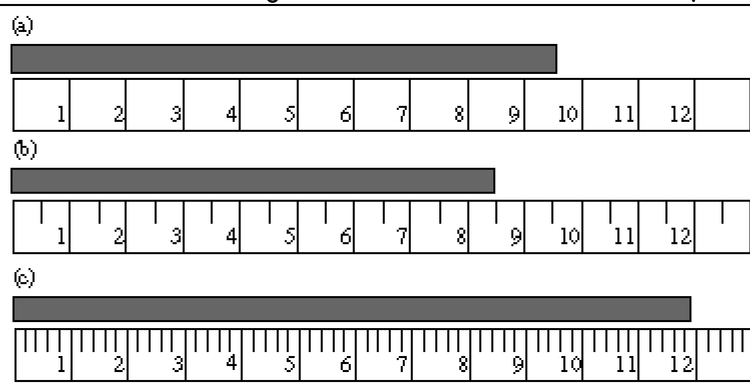


Figure 1 For each object and scale above, determine the least count of the scale, the ILE, and the length of the gray rod. The scales are all in centimeters.

	Least Count (cm)	ILE (cm)	Length (cm)
(a)	1	0.2	9.6
(b)	0.5	0.1	8.5
(c)	0.2	0.05	11.90

(b) Estimated Uncertainty

Often other uncertainties are larger than the ILE. We may try to balance a simple beam balance with masses that have an ILE of 0.01 grams, but find that we can vary the mass on one pan by as much as 3 grams without seeing a change in the indicator. We would use half of this as the **estimated uncertainty**, thus getting uncertainty of ± 1.5 grams.

Another good example is determining the focal length of a lens by measuring the distance from the lens to the screen. The ILE may be 0.1 cm, however the depth of field may be such that the image remains in focus while we move the screen by 1.6 cm. In this case the estimated uncertainty would be half the range or ±0.8 cm.

> **Problem:** I measure your height while you are standing by using a tape measure with ILE of 0.5 mm. Estimate the uncertainty. Include the effects of not knowing whether you are "standing straight" or slouching.

Estimating Uncertainty: Answer

Problem: I measure your height while you are standing by using a tape measure with ILE of 0.5 mm. Estimate the uncertainty. Include the effects of not knowing whether you are "standing straight" or slouching.
There are many possible correct answers to this. However the answer
Δh = 0.5 mm is certainly wrong Here are some of the problems in measuring.

1. As you stand, your height keeps changing. You breath in and out, shift from one leg to another, stand straight or slouch, etc. I bet this would make your height uncertain to at least 1.0 cm.
2. Even if you do stand straight, and don't breath, I will have difficulty measuring your height. The top of your head will be some horizontal distance from the tape measure, making it hard to measure your height. I could put a book on your head, but then I need to determine if the book is level.

I would put an uncertainty of 1 cm for a measurement of your height.

(c) Average Deviation: Estimated Uncertainty by Repeated Measurements

The statistical method for finding a value with its uncertainty is to repeat the measurement several times, find the **average**, and find either the **average deviation** or the **standard deviation**.

Suppose we repeat a measurement several times and record the different values. We can then find the **average** value, here denoted by a symbol between angle brackets, <t>, and use it as our best estimate of the reading. How can we determine the uncertainty? Let us use the following data as an example. Column 1 shows a time in seconds.

Table 1. Values showing the determination of average, average deviation, and standard deviation in a measurement of time. Notice that to get a non-zero average deviation we must take the absolute value of the deviation.

Time, t, sec.	(t - <t>), sec	|t - <t>|, sec	$\left(t-< t >\right)^2$
7.4	-0.2	0.2	0.04
8.1	0.5	0.5	0.25
7.9	0.3	0.3	0.09
7.0	-0.6	0.6	0.36
<t> = 7.6	<t-<t>>= 0.0	<|t-<t>|>= 0.4	$\left\langle\left(t-< t >\right)^2\right\rangle =$ 0.247 Std. dev = 0.50

A simple **average** of the times is the sum of all values (7.4+8.1+7.9+7.0) divided by the number of readings (4), which is 7.6 sec. We will use angular brackets around a symbol to indicate average; an alternate notation uses a bar is placed over the symbol.

Column 2 of Table 1 shows the deviation of each time from the average, (t-<t>). A simple average of these is zero, and does not give any new information.

To get a non-zero estimate of deviation we take the average of the absolute values of the deviations, as shown in Column 3 of Table 1. We will call this the **average deviation**, **Δt**.

Column 4 has the squares of the deviations from Column 2, making the answers all positive. The sum of the squares is divided by 3, (one less than the number of readings), and the square root is taken to produce the sample **standard deviation**. An explanation of why we divide by (N-1) rather than N is found in any statistics text. The sample standard deviation is slightly different than the average deviation, but either one gives a measure of the variation in the data.

If you use a spreadsheet such as Excel there are built-in functions that help you to find these quantities. These are the Excel functions.

=SUM(A2:A5)	Find the sum of values in the range of cells A2 to A5.
=COUNT(A2:A5)	Count the number of numbers in the range of cells A2 to A5.
=AVERAGE(A2:A5)	Find the average of the numbers in the range of cells A2 to A5.
=AVEDEV(A2:A5)	Find the average deviation of the numbers in the range of cells A2 to A5.
=STDEV(A2:A5)	Find the sample standard deviation of the numbers in the range of cells A2 to A5.

For a second example, consider a measurement of length shown in Table 2. The average and average deviation are shown at the bottom of the table.

Table 2. Example of finding an average length and an average deviation in length. The values in the table have an **excess of significant figures**. Results should be rounded as explained in the text.Results can be reported as (15.5 ± 0.1) m or (15.47 ± 0.13) m. If you use standard deviation the length is (15.5 ± 0.2) m or (15.47 ± 0.18) m.

Length, x, m	$\lvert x- \langle x\rangle \rvert$, m	$(x-\langle x\rangle)^2$
15.4	0.06667	0.004445
15.2	0.26667	0.071112
15.6	0.13333	0.017777
15.7	0.23333	0.054443
15.5	0.03333	0.001111
15.4	0.06667	0.004445
Average 15.46667 m	±0.133333 m	St. dev. ±0.17512

We round the uncertainty to one or two significant figures (more on rounding in Section 7), and round the average to the same number of digits relative to the decimal point. Thus the average length with average deviation is either (15.47 ± 0.13) m or (15.5 ± 0.1) m. If we use standard deviation we report the average length as (15.47±0.18) m or (15.5±0.2) m.

Follow your instructor's instructions on whether to use average or standard deviation in your reports.

Problem Find the average, and average deviation for the following data on the length of a pen, L. We have 5 measurements,
(12.2, 12.5, 11.9,12.3, 12.2) cm.

Problem: Find the average and the average deviation of the following measurements of a mass.
(4.32, 4.35, 4.31, 4.36, 4.37, 4.34) grams.

Average Example 1

Problem Find the average, and average deviation for the following data on the length of a pen, L. We have 5 measurements,
(12.2, 12.5, 11.9,12.3, 12.2) cm.

Length (cm)	$\lvert L - L_{ave} \rvert$	$(L - L_{ave})^2$
12.2	0.02	0.0004
12.5	0.28	0.0784
11.9	0.32	0.1024
12.3	0.08	0.0064
12.2	0.02	0.0004
Sum 61.1	Sum 0.72	Sum 0.1880
Average 61.1/5 = 12.22	Average 0.14	$\sigma = \sqrt{\frac{0.1880}{4}} = 0.22$

To get the average sum the values and divide by the number of measurements.

To get the average deviation,

- Find the deviations, the absolute values of the quantity (value minus the average), $\lvert L - L_{ave} \rvert$
- Sum the absolute deviations,
- Get the average absolute deviation by dividing by the number of measurements

To get the standard deviation

- Find the deviations and square them
- Sum the squares
- Divide by (N-1), the number of measurements minus 1 (here it is 4)
- Take the square root.

The pen has a length of (12.22 +/- 0.14) cm or (12.2 +/- 0.1) cm [using average deviations] or (12.22 +/- 0.22) cm or (12.2 +/- 0.2) cm [using standard deviations].

Average Example 2

Problem: Find the average and the average deviation of the following measurements of a mass.
(4.32, 4.35, 4.31, 4.36, 4.37, 4.34) grams.

Mass (grams)	$\lvert m - m_{ave} \rvert$	$(m - m_{ave})^2$
4.32	0.0217	0.000471
4.35	0.0083	0.000069
4.31	0.0317	0.001005
4.36	0.0183	0.000335
4.37	0.0283	0.000801
4.34	0.0017	0.000003
Sum 26.05	0.1100	0.002684
Average 4.3417	Average 0.022	$\sigma = 0.023$

The same rules as Example 1 are applied. This time there are N = 6 measurements, so for the standard deviation we divide by (N-1) = 5.

The mass is (4.342 +/- 0.022) g or (4.34 +/- 0.02) g [using average deviations] or (4.342 +/- 0.023) g or (4.34 +/- 0.02) g [using standard deviations].

(d) Conflicts in the above

In some cases we will get an ILE, an estimated uncertainty, and an average deviation and we will find different values for each of these. We will be pessimistic and **take the largest of the three values as our uncertainty**. [When you take a statistics course you should learn a more correct approach involving adding the variances.] For example we might measure a mass required to produce standing waves in a string with an ILE of 0.01 grams and an estimated uncertainty of 2 grams. We use 2 grams as our uncertainty.

The proper way to write the answer is

- Choose the largest of (i) ILE, (ii) estimated uncertainty, and (iii) average or standard deviation.
- Round off the uncertainty to 1 or 2 significant figures.
- Round off the answer so it has the same number of digits before or after the decimal point as the answer.
- Put the answer and its uncertainty in parentheses, then put the power of 10 and unit outside the parentheses.

Problem: I make several measurements on the mass of an object. The balance has an ILE of 0.02 grams. The average mass is 12.14286 grams; the average deviation is 0.07313 grams. What is the correct way to write the mass of the object including its uncertainty? **What is the mistake in each incorrect one?**
12.14286 g
(12.14 ± 0.02) g
12.14286 g ± 0.07313
12.143 ± 0.073 g
(12.143 ± 0.073) g
(12.14 ± 0.07)
(12.1 ± 0.1) g
12.14 g ± 0.07 g
(12.14 ± 0.07) g

Resolving Conflicts in Different Values of Uncertainty

Problem: I make several measurements on the mass of an object. The balance has an ILE of 0.02 grams. The average mass is 12.14286 grams; the average deviation is 0.07313 grams. What is the correct way to write the mass of the object including its uncertainty? What is the mistake in each one that is incorrect? Go to entire or click on a choice.

1. 12.14286 g	Way wrong! You need the uncertainty reported with the answer. Also the answer has not been properly rounded off.
2. (12.14 ± 0.02) g	Way wrong! You could not read my writing perhaps. The uncertainty is 0.07 grams. Otherwise the format of the answer is fine.
3. 12.14286 g ± 0.07313	Way wrong! You need to round off the uncertainty and the answer. Also the answer should be presented within parentheses.
4. 12.143 ± 0.073 g	Almost there. Put parentheses around the numbers and it would be OK. Rounding off one more place is better.
5. (12.143 ± 0.073) g	This is fine. Slightly better would be to round off one more place.
6. (12.14 ± 0.07)	Almost there, but what pray tell are the units?
7. (12.1 ± 0.1) g	Wrong. You went overboard in rounding. Stop when the uncertainty is 0.07, one significant figure.
8. 12.14 g ± 0.07 g	Almost right. The answer and uncertainty should be in parentheses with unit outside.
9. (12.14 ± 0.07) g	Correct!

The object has a mass of (12.14 g ± 0.07) g. This is the most correct.

Problem: I measure a length with a meter stick with a least count of 1 mm. I measure the length 5 times with results (in mm) of 123, 123, 123, 123, 123. What is the average length and the uncertainty in length?

Problem: I measure a length with a meter stick with a least count of 1 mm. I measure the length 5 times with results in mm of 123, 123, 123, 123, 123. What is the average length and the uncertainty in length?

Length, L (mm)	$\left\lvert L - L_{ave} \right\rvert$	$\left(L - L_{ave}\right)^2$
123	0.0	0.0
123	0.0	0.0
123	0.0	0.0
123	0.0	0.0
123	0.0	0.0
Sum 616	Sum 0.0	Sum 0.0
Average 123	Average 0.0	St. Dev. 0.0

Here the average deviation and the standard deviation are smaller than the ILE of 0.5 mm. Hence I use 0.5 mm as the uncertainty.
The object has a length of (123.0 +/- 0.5) mm.

(e) Why make many measurements? Standard Error in the Mean.

We know that by making several measurements (4 or 5) we should be more likely to get a good average value for what we are measuring. Is there any point to measuring a quantity more often than this? When you take a statistics course you will learn that the **standard error in the mean** is affected by the number of measurements made.

The standard error in the mean in the simplest case is defined as the standard deviation divided by the square root of the number of measurements.

The following example illustrates this in its simplest form. I am measuring the length of an object. Notice that the average and standard deviation do not change much as the number of measurements change, but that the standard error does dramatically decrease as N increases.

Finding Standard Error in the Mean			
Number of Measurements, N	Average	Standard Deviation	Standard Error
5	15.52 cm	1.33 cm	0.59 cm
25	15.46 cm	1.28 cm	0.26 cm
625	15.49 cm	1.31 cm	0.05 cm
10000	15.49 cm	1.31 cm	0.013 cm

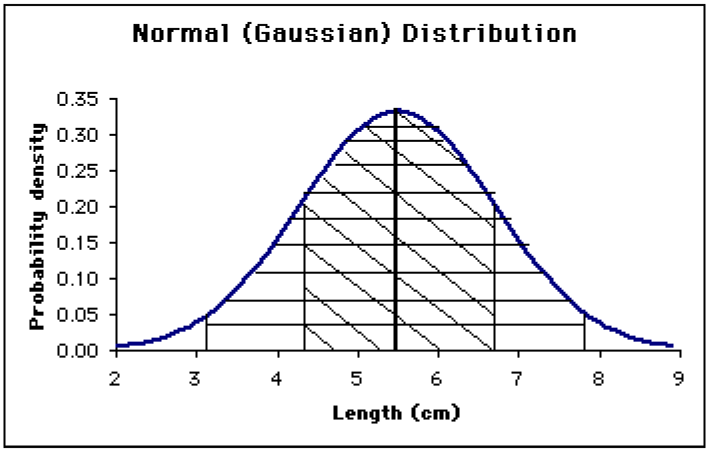
For this introductory course we will not worry about the standard error, but only use the standard deviation, or estimates of the uncertainty.

What is the range of possible values?

When you see a number reported as (7.6 ± 0.4) sec your first thought might be that all the readings lie between 7.2 sec (=7.6-0.4) and 8.0 sec (=7.6+0.4). A quick look at the data in the Table 1 shows that this is not the case: only 2 of the 4 readings are in this range. Statistically we expect 68% of the values to lie in the range of <x> ± Δx, but that 95% lie within <x> ± 2 Δx. In the first example all the data lie between 6.8 (= 7.6 - 2*0.4) and 8.4 (= 7.6 + 2*0.4) sec. In the second example, 5 of the 6 values lie within two deviations of the average. **As a rule of thumb for this course we usually expect the actual value of a measurement to lie within two deviations of the mean.** If you take a statistics course you will talk about **confidence levels**.

How do we use the uncertainty? Suppose you measure the density of calcite as (2.65 ± 0.04) g / cm^3. The textbook value is 2.71 g / cm^3. Do the two values agree? Since the text value is within the range of two deviations from the average value you measure you claim that your value agrees with the text. If you had measured the density to be (2.65 ± 0.01) g / cm^3 you would be forced to admit your value disagrees with the text value.

The drawing below shows a **Normal Distribution** (also called a **Gaussian**). The vertical axis represents the fraction of measurements that have a given value z. The most likely value is the average, in this case <z> = 5.5 cm. The standard deviation is σ = 1.2. The central shaded region is the area under the curve between (<x> - σ) and (x + σ), and roughly 67% of the time a measurement will be in this range. The wider shaded region represents (<x> - 2σ) and (x + 2σ), and 95% of the measurements will be in this range. A statistics course will go into much more detail about this.



Problem: You measure a time to have a value of (9.22 ± 0.09) s. Your friend measures the time to be (9.385 ± 0.002) s. The accepted value of the time is 9.37 s. Does your time agree with the accepted? Does your friend's time agree with the accepted?

Problem: You measure a time to have a value of (9.22 ± 0.09) s. Your friend measures the time to be (9.385 ± 0.002) s. The accepted value of the time is 9.37 s. Does your time agree with the accepted? Does your friend's time agree with the accepted?

We look within 2 deviations of your value, that is between 9.22 - 2(0.09) = 9.04 s and 9.22 + 2(0.09) = 9.40 s. The accepted value is within this range of 9.04 to 9.40 s, so **your experiment agrees with the accepted.**

The news is not so good for your friend. 9.385 - 2(0.002) = 9.381 s and 9.385 + 2(0.002) = 9.389 s. The range of answers for your friend, 9.381 to 9.389 s, does not include the accepted value, so **your friend's time does not agree with the accepted value.**

Problem: Are the following numbers equal within the expected range of values?
(i) (3.42 ± 0.04) m/s and 3.48 m/s?
(ii) (13.106 ± 0.014) grams and 13.206 grams?
(iii) (2.95 ± 0.03) x 10^8 m/s and 3.00 x 10^8 m/s

Problem: Are the following numbers equal within the expected range of values?

(i) (3.42 ± 0.04) m/s and 3.48 m/s?
The 2-deviation range is 3.34 to 3.50 m/s. **Yes the numbers are equal.**

(ii) (13.106 ± 0.014) grams and 13.206 grams?
The 2-deviation range is 13.078 to 13.134 grams. **No the numbers are not equal.**

(iii) (2.95 ± 0.03) x 10^8 m/s and 3.00 x 10^8 m/s

The 2-deviation range is 2.89 x 10^8 to 3.01 x 10^8 m/s. **Yes the numbers are equal.**

Absolute and relative uncertainty

* The terms "experimental error" and "experimental uncertainty" are assumed to have the same meaning in these notes.

Suppose a ruler is used to measure the length of a rod. It is difficult to determine length exactly, but we decide on a value somewhere between 10.1 cm and 10.3 cm, written as

$$L = 10.2 \text{ cm} \pm 0.1 \text{ cm}$$

where 10.2 cm is the most likely for the length and the value ± 0.1 is called the **absolute uncertainty** in the measurement. The uncertainty defines a **range** of possible values for the length, so that

$$10.1 \text{ cm} \leq L \leq 10.3 \text{ cm}.$$

We often use **relative uncertainty,** where

$$\text{relative uncertainty} = \frac{\text{absolute uncertainty}}{\text{measured value}},$$

which, for the rod measurement is $\pm$ (0.1 /10.2) = $\pm$0.009 (no units). The relative uncertainty (or "precision" of the measurement) is often quoted as a percentage so that the **percent uncertainty** above is 0.9%.

In the laboratory, the size of the error is usually estimated according to the equipment being used. You need to select a large enough uncertainty such that the "true value" lies within the range of uncertainty *most* of the time. This is not always straightforward but becomes easier with experience.

Combining Errors in Laboratory Results

In experiments where the desired result is calculated from two or more quantities (e.g. speed = distance/time) the errors in each quantity must be combined to give an uncertainty in the final answer. To calculate $z \pm \delta z$, where $z = f(x, y)$ and $\pm \delta x$ and $\pm \delta y$ are the uncertainties in x and y, we could, in principle, calculate z for all values of $x \pm \delta x$, $y \pm \delta y$. A less tedious method is to calculate the maximum value of δz from the equation:

$$\delta z = (\partial f / \partial x)\, \delta x + (\partial f / \partial y)\, \delta y \qquad (1)$$

which for simple mathematical operations reduces to the following forms:

$$z = x + y \quad \text{and} \quad \delta z = \delta x + \delta y \tag{2a}$$
$$z = x - y \quad \text{and} \quad \delta z = \delta x + \delta y \tag{2b}$$
$$z = x \times y \quad \text{and} \quad \frac{\delta z}{z} = \frac{\delta x}{x} + \frac{\delta y}{y} \tag{2c}$$
$$z = \frac{x}{y} \quad \text{and} \quad \frac{\delta z}{z} = \frac{\delta x}{x} + \frac{\delta y}{y} \tag{2d}$$

Examples

1. When the desired result depends on more than two quantities, the error may be calculated by breaking down the algebraic expression term by term. Thus if $a = bc/d$, the maximum error in a is given by:

$$\frac{\delta a}{a} = \frac{\delta b}{b} + \frac{\delta c}{c} + \frac{\delta d}{d}. \tag{3}$$

2. If $z = x^n$, where n can be positive or negative, we differentiate to obtain $\delta z = nx^{n-1} \delta x$ or, as a relative uncertainty,

$$\frac{\delta z}{z} = \frac{|n|\delta x}{x} \tag{4}$$

3. The density of a metal cylinder may be calculated from

$$\rho = \frac{4m}{\pi d^2 l}$$

where the symbols have their usual meaning. The corresponding error equation is

$$\frac{\delta \rho}{\rho} = \frac{\delta m}{m} + 2\frac{\delta d}{d} + \frac{\delta l}{l}.$$

Note that 4 and π do not contribute to the *relative* error since they are constants.

4.

$$y = a \sin\theta;$$

$$\delta y = \delta a \sin\theta + a\cos\theta\,\delta\theta \qquad \text{(using Eq. (1))}$$

$$\text{or} \qquad \frac{\delta y}{y} = \frac{\delta a}{a} + \frac{\cos\theta\,\delta\theta}{\sin\theta}$$

$$\text{or} \qquad \frac{\delta y}{y} = \frac{\delta a}{a} + \cot\theta\,\delta\theta.$$

provided θ and $\delta\ \theta$ are measured in radians.

5.

$y = a\cos\theta;$

$$\delta y = \delta a \cos\theta + a\sin\theta\delta\theta$$

or $$\frac{\delta y}{y} = \frac{\delta a}{a} + \tan\theta\,\delta\theta.$$

6.

$y = \ln x;$

$$\delta y = \frac{1}{x}\delta x.$$

Mean and Standard Error (continue from Chapter 4)

The arithmetic mean $\bar{x}$ of a set of N readings is defined by

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} \tag{5}$$

where x_i is the ith reading, and Σ means "*add up all the individual values of x_i from i = 1 to N*". The mean is the best estimate of the "true value". Repeated measurements generally follow a *normal* or *Gaussian* probability distribution; the probability of occurrence of an individual value x_i may be calculated from

$$P(x_i) = \frac{1}{\sigma_x\sqrt{2\pi}}\exp\left\{\frac{-(x_i - \bar{x})^2}{2\sigma_x^2}\right\}$$

where $\bar{x}$ is the mean and σ_x is the standard deviation, defined by

$$\sigma_x = \sqrt{\frac{\sum(x_i - \bar{x})^2}{N-1}}. \tag{6}$$

The standard deviation is a measure of the deviation of a typical reading from the mean value. It may be shown that 68% of the measurements lie within one standard deviation of $\bar{x}$ and nearly all measurements (95%) are expected to lie within 2 σ_x of the mean. The quantity σ_x^2 is called the *variance*.

It is generally more useful to consider the *standard error of the mean*, $\sigma_{\bar{x}}$. It may be shown that for N measurements, each subject to an error δx_i, the error in the mean is

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}. \tag{7}$$

Experimental results are usually expressed in the form

$$\bar{x} \pm \sigma_{\bar{x}}.$$

In the special case of radioactive decay, the mean square deviation (from Poisson statistics) is given by

$$\sigma = \sqrt{N}$$

where N is the number of counts. Data are recorded in the form

$$N \pm \sqrt{N}.$$

Hence it is necessary to count for at least 10,000 decays to obtain an accuracy of 1%. For an accuracy of 1 in 103, 106 decay events are required.

Significant Figures (continue from Chapter 4)

Suppose you use a calculator to obtain a standard deviation of 0.987654321. If this is larger than the reading error in your measurements, then this will be the error in each of the data points. For a sample of *N* data points, the expected uncertainty in the standard deviation (i.e., the error in the error) is:

$$\delta\sigma_{est} = \frac{\sigma_{est}}{\sqrt{2N-2}}$$

Suppose $\delta\sigma_{est}$ = 0.232792356. This means that the actual value of the standard deviation lies between 0.987654321 - 0.232792356 = 0.704861965, and 0.987654321 + 0.232792356 = 1.220446677. A moment's thought about this should convince you that many of these digits have no significance.

The value of the estimated standard deviation is more like 0.99 $\pm$ 0.23 or maybe even 1.0 $\pm$ 0.2. Writing 0.988 $\pm$ 0.233 has more digits than are actually significant. Even if you repeat a measurement 50 times, the estimated standard deviation has at most only two digits that have any meaning.
Imagine that one of the data points has a numerical value of 12.3456789. If we estimate the standard deviation to be 0.99, then the point value is 12.35 $\pm$ 0.99. It would be wrong to say 12.345 $\pm$ 0.99, since the ‘5’ has no meaning.

In the laboratory, the reading error will be the error in each individual measurement. This will be little more than a guess made by the experimenter, and it is doubtful that you can guess to more than one significant figure. Thus a reading error almost by definition has only **one** significant figure, and that number determines the significant figures in the value itself.

You need to be particularly careful when writing down computer-generated results. A slope of 0.0778465 $\pm$ 0.00217814 is meaningless: the error should written as $\pm$0.002, which means that the slope should be written as 0.078 to keep the same number of figures after the decimal point.

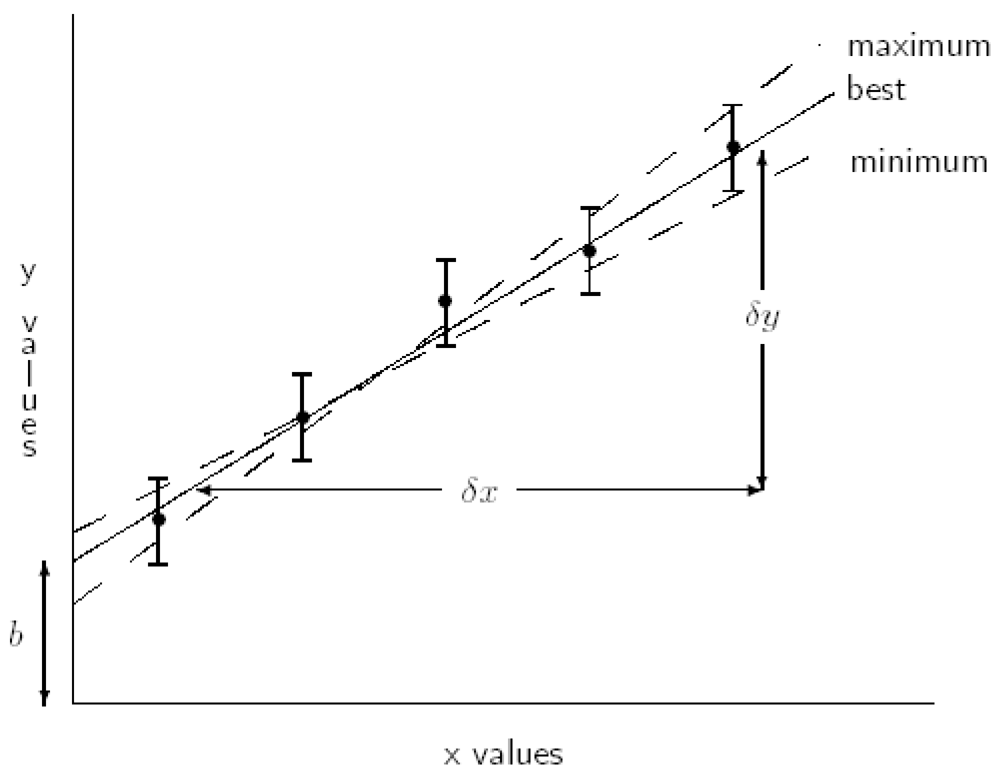
Similarly, if the voltage across a resistor is 15.4 $\pm$ 0.1 volts and the current is 1.7 $\pm$ 0.1 amps, the resistance is **not** 9.0588235 ohms because any additional figures beyond the first decimal place are meaningless and The final value for *R* should be written as (9.1 $\pm$ 0.6) Ω.

The Errors in a Straight Line Graph

Often the result of an experiment depends on the slope of a straight-line graph. Errors associated with the data are illustrated by *error bars*, the size of which defines the range of uncertainty in one (or both) axes. A straight line is represented by the equation

$$y = mx + b$$

where *m* is the slope and *b* is the *y*-intercept.



The uncertainty in the slope is estimated by considering extremes of maximum and minimum slope, which might concievably, fit the data, as illustrated in the diagram. Denoting the slopes of these lines by m_{max} and m_{min} respectively, δm may be calculated from

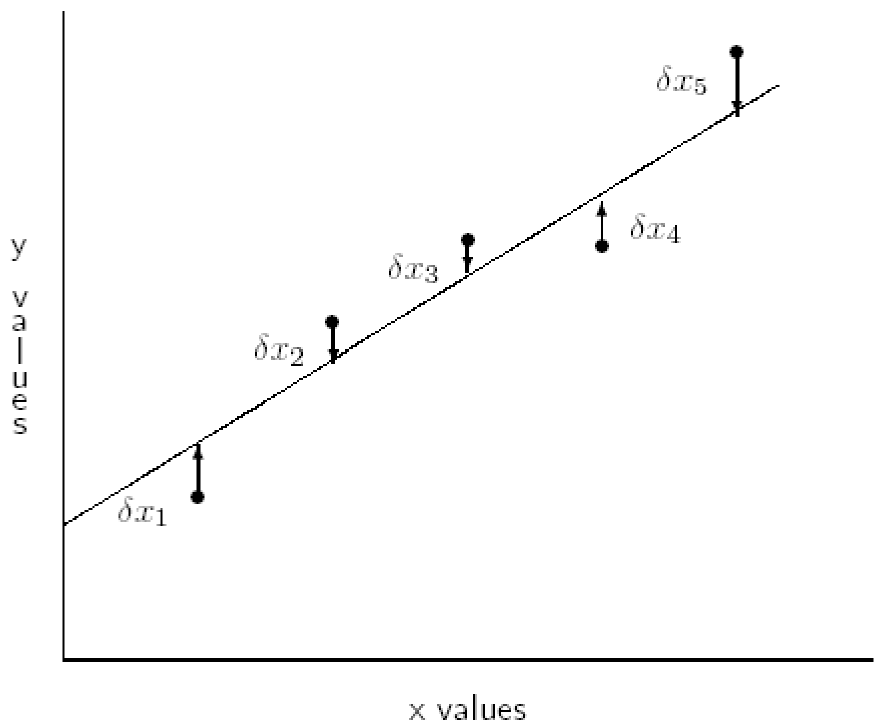
$$\delta m = \frac{m_{max} - m_{min}}{2} \tag{8}$$

and the error in the intercept:

$$\delta b = \frac{b_{max} - b_{min}}{2}. \tag{9}$$

Method of Least Squares

Linear regression by the *Method of Least Squares* finds the equation of the best-fit line by minimizing the sum of the squares of deviations of the individual *y* values from the straight line.



The slope and intercept are given by

$$m = \frac{\sum(xy) - {}^1\!/_{\mathrm{N}}(\sum x)(\sum y)}{\sum(x^2) - {}^1\!/_{\mathrm{N}}(\sum x)^2} \tag{10}$$

and

$$b = {}^1\!/_{\mathrm{N}}\left(\sum y - m \sum x\right) \tag{11}$$

with uncertainties

$$\delta b = \sqrt{\frac{\sum(y^2) - b\sum y - m\sum(xy)}{N-2}} \tag{12}$$

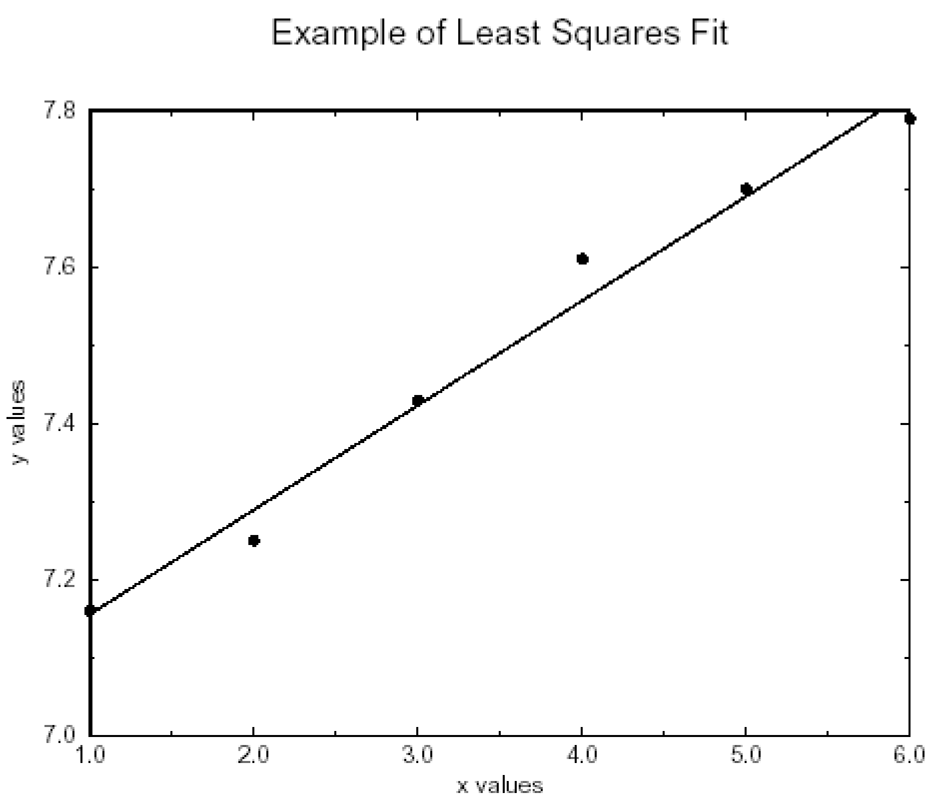
and

$$\delta m = \frac{\delta b}{\sqrt{\sum(x^2) - {}^1\!/\!_N(\sum x)^2}} \tag{13}$$

Worked Example

Linear regression is easily performed by computer, however, a detailed calculation is shown here for completeness.

x =1.0, 2.0, 3.0, 4.0, 5.0, 6.0
y =7.16, 7.25, 7.43, 7.61, 7.70, 7.79,



i	x_i	y_i	x_i^2	y_i^2	$x_i y_i$
1	1.0	7.16	1.0	51.26	7.16
2	2.0	7.25	4.0	52.56	14.5
3	3.0	7.43	9.0	55.21	22.29
4	4.0	7.61	16.0	57.91	30.44
5	5.0	7.70	25.0	59.29	38.5
6	6.0	7.79	36.0	60.68	46.74
sum	21.0	44.94	91.0	336.92	159.63

and hence,

$$\sum x = 21.0; \qquad \sum y = 44.94; \qquad \sum x^2 = 91.0; \qquad \sum y^2 = 336.92$$

and

$$\sum xy = 159.63.$$

Finally we obtain

$$m = \frac{159.63 - 1/6 \times 21.0 \times 44.94}{91.0 - 1/6(21.0)^2} = 0.134$$

and

$$b = 1/6\,(44.94 - 0.134 \times 21.0) = 7.02$$

The corresponding uncertainties is given by

$$\delta b = \sqrt{\frac{336.92 - 7.02 \times 44.94 - 0.134 \times 159.63}{4}} = 0.04$$

and

$$\delta m = \frac{0.04}{\sqrt{91.0 - 1/6(21.0)^2}} = 0.009$$

The relative uncertainty in the slope, therefore, is

$$\frac{0.009}{0.134} = 0.067 \text{ or } 6.7\%$$

and the relative uncertainty in the intercept is

$$\frac{0.04}{7.02} = 0.005 \text{ or } 0.5\%.$$

When the data points do not follow a straight line, the best curve through the points may also be obtained by linear regression provided the function is a polynomial of the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Other mathematical functions may be fitted using *non-linear regression*.

> In most of the lab experiments that you will do, the uncertainty in the slope of the straight line will be greater than the uncertainties in other measured quantities. This means that you can usually ignore the errors in everything but the slope.